Exchangeable Convertible Bond Valuation

A convertible bond issuer pays periodic coupons to the convertible bond holder. The bond holder can convert the bond into the underlying stock within the period(s) of time specified by the conversion schedule. The bond issuer can call the bond and the holder can put it according to the call and put provisions.

The Exchangeable feature assumes that the convertible bond and the underlying stock are issued by different parties. There are two possible cases with respect to stock conversion:

· *covered*: the bond issuer holds the underlying stock for the life time of the convertible;
· *vulnerable*: the bond issuer does not hold the stock prior to the stock conversion.

We assume a convertible bond that is specified by

- $(t_1, t_2, \ldots, t_M)$ - the set of coupon dates, t_M is also the bond maturity date;
- $(c_1, c_2, \ldots, c_M)$ - the set of coupons, each coupon payable on the respective coupon date;
- P - the bond principal, payable at maturity date, t_M.
- T_x - the exercise date on which the bond can be converted into the underlying stock, which pays dividends.

We consider discrete dividends that are specified by absolute amounts paid at pre-determined times. In particular we assume that an absolute amount, d_i, is paid at a time, T_i, for $i = 1,\ldots,N$, where $0 < T_1 < \ldots < T_N \le T$ (here $T \equiv t_M$ is the convertible's maturity). We model the stock price based on a Hull-White approach. Specifically, let

- G_t denote a capital gains process,
- S_t represent the stock price,
- τ denote the bond issuer's default time.

Let

$$D_i = \sum_{j=i}^{N} d_j e^{-\int_0^{T_j} r(s)ds} \quad (1)$$

for $i = 1,\ldots,N$, where r_t is a deterministic risk-free rate. We then assume that

$$S_t = \left(G_t + e^{\int_0^t (r(s)+h(s))ds} \sum_{i=1}^{N} D_i I_{(T_{i-1},T_i]}(t) \right) 1_{\tau > t}, \quad (2)$$

for $t > 0$, where

- h_t is a deterministic hazard-rate,
- $I_{(T_{i-1},T_i]}(t) = \begin{cases} 1, & t \in (T_{i-1}, T_i], \\ 0, & \text{otherwise}, \end{cases}$

(here we set $T_0 = 0$).

Furthermore the process $\{G_t | t \geq 0\}$ satisfies, under the risk-neutral probability measure, an SDE of the form

$$dG_t = G_t([r_t + h_t]dt + \sigma dW_t) \quad (3)$$

where

- σ is a constant volatility parameter,
- $\{W_t | t \geq 0\}$ is a standard Brownian motion,
- $G_0 = S_0 - \sum_{i=1}^{N} d_i e^{-\int_0^{T_i} r(s)ds}$. (4)

Thus,

$$G_t = G_0 e^{\int_0^t (r+h)ds - \frac{\sigma^2}{2}t + \sigma W} \quad (5)$$

Let

- T_j be the dividend date immediately after the option maturity date, i.e. $T_j = \min(T_1, T_2, \ldots, T_N)$ such that $T_j \geq T_x$, where $(T_1, \ldots, T_N)$ is the set of dividend payment dates;
- t_m is the coupon date immediately after the option maturity date, i.e. $t_m = \min(t_1, t_2, \ldots, t_M)$ such that $t_m \geq T_x$.

Then

$$S_{T_x} = \left(G_{T_x} + e^{\int_0^{T_x}(r+h)ds} \sum_{i=j}^{N} d_i\, e^{-\int_0^{T_i} r ds}\right) 1_{\tau > T_x} = e^{\int_0^{T_x}(r+h)ds} \left(G_0 e^{-\frac{\sigma^2}{2}T_x + \sigma W_{T_x}} + \sum_{i=j}^{N} d_i\, e^{-\int_0^{T_i} r ds}\right) 1_{\tau > T_x} \quad (6)$$

Using arguments similar to those used in ref [1] one can show that the T_x-time price of the straight bond (see https://finpricing.com/lib/FiZeroBond.html), B_{T_x}, is

$$B_{T_x} = \left(Pe^{-\int_{T_x}^{T}(r+h)ds} + \sum_{k=m}^{M} c_k e^{-\int_{T_x}^{t_k}(r+h)ds} \right) \mathbf{1}_{\tau > T_x} \qquad (7)$$

and its present value is

$$B_{0,T_x} = Pe^{-\int_{0}^{T}(r+h)ds} + \sum_{k=m}^{M} c_k e^{-\int_{0}^{t_k}(r+h)ds} \qquad (8)$$

If the conversion ratio is γ, the present value of the conversion option is $V =$
$E\left(e^{-\int_0^{T_x} rds}\, max\left(\gamma S_{T_x} - B_{T_x}, 0\right) \middle| \mathfrak{F}_0\right) =$

$$= E\left[max\left(e^{\int_0^{T_x} hds} \gamma \left(G_0 e^{-\frac{\sigma^2}{2}t + \sigma W} + \sum_{i=j}^{N} d_i\, e^{-\int_0^{T_i} rds} \right) 1_{\tau > T_x} \right.\right.$$

$$\left.\left. - \left(Pe^{-\int_0^{T} rds} e^{-\int_{T_x}^{T} hds} + \sum_{k=m}^{M} c_k e^{-\int_0^{t_k} rds} e^{-\int_{T_x}^{t_k} hds} \right) 1_{\tau > T_x}, 0 \right) | \mathfrak{F}_0 \right] =$$

$$= e^{\int_0^{T_x} hds} E\left[max\left(\left(\left(\gamma G_0 e^{-\frac{\sigma^2}{2}t + \sigma W} + \gamma \sum_{i=j}^{N} d_i\, e^{-\int_0^{T_i} rds} - Pe^{-\int_0^{T}(r+h)ds} \right.\right.\right.\right.$$

$$\left.\left.\left.\left. - \sum_{k=m}^{M} c_k e^{-\int_0^{t_k}(r+h)ds} \right) 1_{\tau > T_x}, 0 \right) | \mathfrak{F}_0 \right]$$

Using the assumption that the time to default process is independent from the Brownian motion process, W_t, that drives the capital gain process (see ref[1]), and the fact that the probability of default before time T_x, $P(\tau > T_x) = exp\left(-\int_0^{T_x} hds\right)$, we get further

$$V = E\left[max\left(\left(\gamma G_0 e^{-\frac{\sigma^2}{2}t+\sigma W} - K\right), 0\right) | \mathfrak{F}_0\right] \quad (9)$$

where

$$K = Pe^{-\int_0^T (r+h)ds} + \sum_{k=m}^{M} c_k e^{-\int_0^{t_k}(r+h)ds} - \gamma \sum_{i=j}^{N} d_i\, e^{-\int_0^{T_i} rds} \equiv B_{0,T_x} - \gamma PV(remngDvd) \quad (10)$$

Finally, one gets

$$V = SN(d_1) - KN(d_2) \quad (11)$$

where

$$S = \gamma G_0, \quad (12)$$

$$d_1 = \frac{\ln \frac{S}{K} + \sigma^2 T_x}{\sigma\sqrt{T_x}}, \quad (13)$$

$$d_2 = \frac{\ln \frac{S}{K} - \sigma^2 T_x}{\sigma\sqrt{T_x}}, \quad (14)$$

and $N(g)$ is the standard normal cumulative distribution function.

The convertible bond price is

$$CB = V + B_{0,T_x} \tag{15}$$

where V is given by eq. (9) and B_{0,T_x} by eq. (8).