CMS Spread Option Valuation

A constant maturity swap (CMS) spread option makes payments based on a bounded spread between two index rates (e.g., a GBP CMS rate and a EURO CMS rate). The GBP CMS rate is calculated from a 15 year swap with semi-annual, upfront payments, while the EURO CMS rate is based on a 15 year swap with annual, upfront payments.

Let

- T be a CMS rate fixing time,
- $T+\Delta$ be a payment time,
- G and E respectively denote the GBP and EURO CMS rates at the fixing time,
- $S = 2G - E$ be a CMS rate spread.

Then, at time $T+\Delta$, the buyer must pay a spread, S, bounded below and above by 0% and 10% respectively, multiplied by the accrual period, Δ; formally the payoff is given by

$$\Delta \min\left(10\%, max\left(0\%, S\right)\right).$$

We assume that both the forward GBP and EURO CMS rates follow geometric Brownian motion under their respective T-forward measures. Here respective initial forward CMS rates are calculated. The forward rates are then convexity adjusted from respective parallel bonds specified using

- the same payment times as the underlying CMS,

- coupon equal to the forward CMS rate,
- initial bond yield equal to the forward CMS rate.

Let

- G_t be a GBP swap rate at time t,
- U_t be a EURO swap rate at time t,
- $S_T = 2G_T - U_T$ be a spread between the two swap rates above.

Then

$$\begin{aligned}
\min(\max(S_T,0),10\%) &= -\max(-\max(S_T,0),-10\%),\\
&= -(\max(0,\max(S_T,0)-10\%)-\max(S_T,0)),\\
&= \max(S_T,0)-\max(0,\max(S_T,0)-10\%),\\
&= \max(S_T,0)-(-10\%+\max(10\%,\max(S_T,0))),\\
&= \max(S_T,0)+10\%-\max(10\%,\max(S_T,0)),\\
&= \max(S_T,0)+10\%-\max(10\%,S_T),\\
&= \max(S_T,0)+10\%-(10\%+\max(0,S_T-10\%)),\\
&= \max(S_T,0)-\max(0,S_T-10\%).
\end{aligned}$$

Furthermore, since $S_T = \max(S_T,0)-\max(-S_T,0)$,

$$\min(\max(S_T,0),10\%) = S_T + \max(-S_T,0) - \max(0,S_T-10\%).$$

The buyer must pay ΔS_T at time $T+\Delta$ where Δ (with $\Delta > 0$) is an accrual period. Let

- $E^{T+\Delta}$ denote the domestic $T+\Delta$-forward probability measure,

- E^T denote the domestic T-forward probability measure,
- E^Q denote the domestic risk-neutral probability measure.

Then

$$
\begin{aligned}
P(0,T+\Delta)E^{T+\Delta}(S_T) &= E^Q\left(\frac{S_T}{\beta_{T+\Delta}}\right), \\
&= E^Q\left(E^Q\left(\frac{S_T}{\beta_{T+\Delta}}\middle|F_T\right)\right), \\
&= E^Q\left(S_T E^Q\left(\frac{1}{\beta_{T+\Delta}}\middle|F_T\right)\right), \\
&= E^Q\left(\frac{S_T P(T,T+\Delta)}{\beta_T}\right), \\
&= P(0,T)E^T\left(S_T P(T,T+\Delta)\right),
\end{aligned}
$$

where $P(t,\tau)$ is the price at time t of a zero-coupon bond that matures at time τ with face value of 1 Euro.

We assume that the forward CMS rate processes, $\{G_t \ |0 \le t \le T\}$ and $\{U_t \ |0 \le t \le T\}$, satisfy, under domestic T-forward measure, the respective SDEs

$$dG_t = G_t \sigma_G dW_t^G,$$

$$dU_t = U_t \sigma_U dW_t^U,$$

where

- σ_G and σ_U are respective constant volatilities,
- $\{W_t^G \; | t \geq 0\}$ and $\{W_t^U \; | t \geq 0\}$ are standard Brownian motions with instantaneous correlation, ρ_{GU}.

Here G_o is determined from respective convexity and quanto adjustments, while U_o is determined from a convexity adjustment.

Let y_t satisfy

$$\frac{1}{1+\Delta y_t} = \frac{P(t,T+\Delta)}{P(t,T)}.$$

Observe that, under domestic T-forward probability measure,

$$\begin{aligned} E^T\left(\frac{1}{1+\Delta y_t}\middle| F_\tau\right) &= E^T\left(\frac{P(t,T+\Delta)}{P(t,T)}\middle| F_\tau\right), \\ &= \frac{P(\tau,T+\Delta)}{P(\tau,T)}, \\ &= \frac{1}{1+\Delta y_\tau}, \end{aligned}$$

so that $\left\{\frac{1}{1+\Delta y_t}\middle| 0 \leq t \leq T\right\}$ is a martingale under this measure. We assume that the process $\{y_t | 0 \leq t \leq T\}$ satisfies, under domestic T-forward measure, the SDE

$$dy_t = y_t \sigma_y dW_t^y$$

where

- σ_y is a constant volatility,
- $\{W_t^Y | 0 \le t \le T\}$ is a standard Brownian motion with respective pairwise correlation to $\{W_t^U | 0 \le t \le T\}$ and $\{W_t^G | 0 \le t \le T\}$, ρ_{YU} and ρ_{YG},
- $\dfrac{1}{1+\Delta y_0} = \dfrac{P(0,T+\Delta)}{P(0,T)}$.

Assuming that $\{y_t | 0 \le t \le T\}$ satisfies the SDE above, however, the martingale property above does not hold.

In order to satisfy the martingale condition

$$E^T\left(\frac{1}{1+\Delta y_T}\right) = \frac{P(0,T+\Delta)}{P(0,T)},$$

we convexity adjust the initial forward bond yield, y_0.

Observe that

$$X_0 P^f(0,T) E^{T_f}\left(\frac{G_T}{1+\Delta y_T^f}\right) = P^d(0,T) E^{T_d}\left(\frac{G_T X_T}{1+\Delta y_T^f}\right),$$

$$= P^d(0,T) E^{T_d}\left(\frac{G_T x_T}{1+\Delta y_T^d}\right),$$

where

- X_t is the spot exchange rate from foreign (GBP) currency into domestic (EURO) currency,
- x_t is the forward exchange rate, at seen at time t, from foreign currency into domestic, at the forward time $T+\Delta$.

From the above the correlation, under foreign T-forward measure, between the Brownian motions respectively driving the foreign bond yield and GBP swap rate processes constrains the correlation between the Brownian motions driving the Euro bond yield and forward GBP CMS rate processes under domestic T-forward measure. Bond yield is computed from bond price (see https://finpricing.com/lib/FiZeroBond.html).

To evaluate the expression above, we require the SDE followed by the forward exchange rate process under domestic T-forward measure. This is derived by changing measure from foreign $T+\Delta$-forward measure, to domestic risk-neutral measure, to domestic T-forward measure.